

On the Frame-Like Formulation of Mixed-Symmetry Massless Fields in $(A)dS_d$

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Abstract

The frame-like covariant Lagrangian formulation of bosonic and fermionic mixed-symmetry type higher spin massless fields propagating on the AdS_d background is proposed. Higher spin fields are described in terms of gauge p -forms which carry tangent indices representing certain traceless tensor or gamma transversal spinor-tensor representations of the AdS_d algebra $o(d-1, 2)$ (or $o(d, 1)$ for bosonic fields in dS_d). Manifestly gauge invariant Abelian higher spin field strengths are introduced for the general case. We describe the general framework and demonstrate how it works for the mixed-symmetry type fields associated with the three-cell “hook” and arbitrary two-row rectangular tableaux. The manifestly gauge invariant actions for these fields are presented in a simple form. The flat limit is also analyzed.

1 Introduction

The problem of covariant Lagrangian description of arbitrary spin fields propagating on flat [1]-[17] and (anti)de Sitter $((A)dS)$ [18]-[30] backgrounds attracts considerable attention. The interest is motivated by the fact that higher spin fields and higher spin symmetries show up in a wide range of models from string theories to higher spin gauge theories describing interacting dynamics of massless fields on the Minkowski [31]-[35] and the $(A)dS_d$ backgrounds [36]-[42] (for review and more references see [41, 42]). Possible relations of higher spin gauge theory with a tensionless limit of string theory in AdS space and boundary conformal models was extensively discussed in [43]-[60] in the context of weak coupling regime of the AdS/CFT correspondence [61]-[63].

To date, symmetric higher spin field dynamics (both massive and massless) provides the most elaborated case among the variety of unitary irreps of Poincare and AdS algebras [2, 3, 6, 18, 19, 20, 21, 24]. To some extent, this is because in the four dimensional space-time there is no room for mixed-symmetry irreps except for dual theories involving “exotic” symmetry type dynamical variables¹. However, for higher space-time dimensions, mixed-symmetry representations do appear and the problem of their field-theoretical description has not been yet worked out in full generality. In case of Minkowski space several approaches were suggested to analyze mixed-symmetry fields [12, 13, 15, 17]. Covariant formulation for generic mixed-symmetry fields in AdS_d is still lacking however, despite some progress achieved in [23, 26, 27, 29, 30]. The peculiarity which complicates the straightforward extension of the flat results is that the classification of massless fields is essentially different for Poincare and AdS_d algebras. From the field-theoretical perspective, this fact manifests itself in different sets of gauge symmetries in flat and AdS_d space-times [22, 23]. As a consequence, an irreducible AdS_d mixed-symmetry field decomposes into a set of flat fields in the flat limit [23].

In this paper we propose a new approach to the covariant description of generic mixed-symmetry fields propagating on the AdS_d background, which generalizes the “gauge” formulation of the symmetric field dynamics developed previously in [6, 19, 20, 21, 39] as well as analogous first-order approach elaborated by Zinoviev in [30] for particular mixed-symmetry fields.

The construction is surprisingly simple. Let a lowest weight unitary massless representation of the AdS_d algebra $o(d-1, 2)$ be characterized by the lowest energy subspace described as a representation of $o(d-1) \subset o(d-1, 2)$ by a traceless Young tableau $Y_{o(d-1)}$ which has a longest row of length s and a shortest column of height p . Then the corresponding field-theoretical system can be described by a p -form gauge field which takes values in the representation of the AdS_d algebra $o(d-1, 2)$ described by the traceless Young tableau $Y_{o(d-1,2)}$ obtained from that of $Y_{o(d-1)}$ by cutting the shortest column and adding the longest row of length $s-1$. The resulting p -form gauge field contains the physical higher spin gauge field along with all necessary auxiliary and extra fields and allows one to construct manifestly gauge invariant field strengths to be used to build invariant action in the MacDowell-Mansouri form [68]. The formulation in terms of p -form connections and higher spin curvatures allows us to control higher spin gauge symmetries in geometric terms.

As far as bosonic massless fields are concerned, the proposed formulation works equally well in de Sitter background. For fermions this is not the case because the dS reality conditions for massless fields require imaginary mass-like parameters in the action. For definiteness we will mostly refer to the AdS case in this paper.

The paper is organized as follows. In section 2 we present the general scheme, fix an appropriate set of fields and gauge symmetries, discuss a form of the action functional and generalized Weyl tensors. The particular examples of three-cell “hook” tableau, four-cell “window” tableau, and an arbitrary two-row rectangular Young tableau are considered, respectively, in subsections 3.1, 3.2 and 3.3 of section

¹For detailed discussion of dual theories in diverse dimensions see recent papers [64], [15], [16], [65], [66], [67].

3. For these models we build manifestly gauge invariant actions which properly describe the field dynamics on the $(A)dS_d$ background and investigate their flat limits. Conclusion is given in section 4.

2 General scheme

2.1 Young tableaux and trace conditions

Let $A^{(s_1, \dots, s_q)}$ denote a tensor²

$$A^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)} , \quad (2.1)$$

which is symmetric in each group of indices³ $a_i(s_i)$ and satisfies the Young symmetry conditions associated with the Young tableau $Y(s_1, s_2, \dots, s_q)$ composed of q rows of lengths $s_1 \geq s_2 \geq s_3 \geq \dots \geq s_q > 0$, *i.e.* symmetrization of all indices in i^{th} row with any index from some $(i+k)^{th}$ ($k > 0$) row gives zero.

Let the tensor (2.1) satisfy the conditions

$$\eta_{a_i a_i} \eta_{a_i a_i} A^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)} = 0 , \quad 0 < i \leq m \quad (2.2)$$

and

$$\eta_{a_i a_i} A^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)} = 0 , \quad m < i \leq q , \quad (2.3)$$

where η_{ab} is some metric, which has signature (p, r) (for the Lorentz case, for example, $p = d - 1$ and $r = 1$) and m is some non-negative integer. The condition (2.2) means that contraction of any two pairs of indices from any of the first m rows of the Young tableau gives zero. The condition (2.3) means that contraction of any pair of indices from the last $q - m$ rows gives zero.

The linear space of tensors (2.1) which have the Young properties of the type $Y(s_1, \dots, s_q)$ and satisfy the conditions (2.2), (2.3) will be denoted $B_m^{p,r}(s_1, \dots, s_q)$. Note that $B_i^{p,r}(s_1, \dots, s_q) \subset B_j^{p,r}(s_1, \dots, s_q)$ for $i < j$. $B_0^{p,r}(s_1, \dots, s_q)$ is the space of traceless tensors with the $Y(s_1, \dots, s_q)$ Young properties.

The following lemmas are simple consequences of the definitions (2.2)-(2.3) and the Young symmetry properties of (2.1).

Lemma 1

Contraction of $\eta_{(a_i a_j} \eta_{a_k a_l)}$ with any four symmetrized indices of a tensor from $B_m^{p,r}(s_1, \dots, s_q)$ gives zero.

²Throughout the paper we work within the mostly minus signature and use notations $\underline{m}, \underline{n} = 0 \div d - 1$ for world indices, $a, b = 0 \div d - 1$ for tangent Lorentz $so(d - 1, 1)$ vector indices and $A, B = 0 \div d$ for tangent $(A)dS_d$ ($so(d - 1, 2)$) $so(d, 1)$ vector indices. We also use condensed notations of [19] for a set of symmetric vector indices: $a(k) \equiv (a_1 \dots a_k)$. Upper (lower) indices denoted by the same letter are assumed to be symmetrized as $X^a Y^a \equiv \frac{1}{2!} (X^{a_1} Y^{a_2} + X^{a_2} Y^{a_1})$ prior contractions.

³Usually, the parameter q in (2.1) satisfies the inequality $q \leq \nu$, where $\nu = [\frac{d}{2}] - 1$ is the rank of the little group $SO(d - 2)$ for Minkowski space or $\nu = [\frac{d-1}{2}]$ is the rank of $SO(d - 1)$ in the case of AdS_d , although dual descriptions with larger q are also possible.

Lemma 1 is a corollary of (2.2) and the Young symmetry properties, which guarantee that any group of symmetrized indices can be placed in the first row.

Lemma 2

From Lemma 1 it follows that

$$\eta_{a_i(a_j} \eta_{a_k a_l)} A^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)} = 0, \quad \forall i, j, k, l, \quad (2.4)$$

i.e. any double trace gives zero provided that any three of the contracted indices are symmetrized.

This is because $\eta_{ab}\eta_{cd}$ belongs to the symmetric part of the tensor product

$$\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right)_{\text{sym}} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}. \quad (2.5)$$

Nonzero traces in $B_m^{p,r}(s_1, \dots, s_q)$ therefore can only appear when all elementary contractions hit different rows.

Lemma 3

The condition (2.3) along with Lemma 2 mean that contraction of any $m + 1$ pairs of indices of $A^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)} \in B_m^{p,r}(s_1, \dots, s_q)$ gives zero.

It is convenient to treat Young tableaux as built of horizontal rectangular blocks (s, p) of length s and height p as elementary entities:

$$\begin{array}{c} s \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} p \end{array} \quad (2.6)$$

Then one operates with a Young tableau $Y(s_1, \dots, s_q)$ with indices rearranged into elementary blocks

$$A^{(s_1, \dots, s_q)} \sim A^{(\tilde{s}_1, p_1); (\tilde{s}_2, p_2); \dots; (\tilde{s}_k, p_k)}, \quad (2.7)$$

where blocks are described by the sets of pairs of positive integers (\tilde{s}_i, p_i) with $\tilde{s}_1 > \tilde{s}_2 > \dots > \tilde{s}_k > 0$ and p_i such that $\sum_i p_i = q$. The exact identification in (2.7) is

$$\tilde{s}_1 = \underbrace{s_1 = \dots = s_{p_1}}_{p_1} > \tilde{s}_2 = \underbrace{s_{p_1+1} = \dots = s_{p_1+p_2}}_{p_2} > \dots > \tilde{s}_k = \underbrace{s_{p_1+\dots+p_{k-1}+1} = \dots = s_q}_{p_k}. \quad (2.8)$$

For the upper block it is sometimes convenient to use notations $\tilde{s}_1 = s$ and $p_1 = p$.

Recall that a rectangular block is invariant (may be up to a sign) with respect to exchange of its rows. As a result it follows

Lemma 4

Once (2.3) is true for one of the rows of a rectangular block it is true for the entire block, i.e. $B_{m_1}^{p,r}(s_1, \dots, s_q) = B_{m_2}^{p,r}(s_1, \dots, s_q)$ if $s_{m_1+1} = s_{m_2+1}$.

Therefore, it is sufficient to impose the trace condition (2.3) for any row inside a horizontal block (e.g., upper row).

2.2 Background geometry and compensators

The background Minkowski or $(A)dS_d$ geometry is described by the frame field $h^a = h_{\underline{n}}^a dx^{\underline{n}}$ and Lorentz spin connection $\omega^{ab} = \omega_{\underline{n}}^{ab} dx^{\underline{n}}$ which obey the equation

$$[\mathcal{D}_{\underline{m}}, \mathcal{D}_{\underline{n}}] A^{a_1(s_1), \dots, a_q(s_q)} = \lambda^2 (s_1 h_{\underline{m}}^{a_1} h_{\underline{n}c} A^{a_1(s_1-1)c, \dots, a_q(s_q)} + \dots) - (\underline{m} \leftrightarrow \underline{n}), \quad (2.9)$$

where

$$\mathcal{D}_{\underline{n}} A^{a_1(s_1), \dots, a_q(s_q)} = \partial_{\underline{n}} A^{a_1(s_1), \dots, a_q(s_q)} + s_1 \omega_{\underline{n}}^{a_1}{}_c A^{a_1(s_1-1)c, \dots, a_q(s_q)} + \dots, \quad \partial_{\underline{n}} = \frac{\partial}{\partial x^{\underline{n}}}. \quad (2.10)$$

The zero-torsion condition $\mathcal{D}_{\underline{n}} h_{\underline{m}}^a - \mathcal{D}_{\underline{m}} h_{\underline{n}}^a = 0$ is imposed. It expresses $\omega_{\underline{m}}^{ab}$ in terms of $h_{\underline{m}}^a$. Note that the equation (2.9) describes AdS_d space-time with the symmetry algebra $o(d-1, 2)$ when $\lambda^2 > 0$ and dS_d space-time with the symmetry algebra $o(d, 1)$ when $\lambda^2 < 0$. Minkowski space-time corresponds to $\lambda = 0$. In the fermionic case, massless equations contain mass-like terms expressed in units of λ . A formal complication for the de Sitter case is that these terms become imaginary.

In the sequel, we extensively use $(A)dS_d$ covariant notations and operate with Young tableaux $A^{A_1(s_1), A_2(s_2), \dots, A_k(s_k)}(x)$, where $A_i = 0 \div d$ is an $o(d-1, 2)$ or $o(d, 1)$ vector index. To relate the $(A)dS$ covariant approach with the Lorentz-covariant approach it is useful to introduce the compensator field $V^A(x)$ normalized as $V^A V_A = \pm 1$ [69]. It allows one to identify the Lorentz subalgebra $so(d-1, 1)$ of the $(A)dS_d$ algebra $(so(d-1, 2))so(d, 1)$ with the stability algebra of the compensator. With the help of the compensator field, the covariant splitting of the $(so(d-1, 2))so(d, 1)$ 1-form connection $\Omega^{AB} = -\Omega^{BA}$ into the frame field E^A and the Lorentz spin connection $\omega^{AB} = -\omega^{BA}$ is defined as follows

$$\lambda E^A = DV^A \equiv dV^A + \Omega^{AB} V_B, \quad \omega^{AB} = \Omega^{AB} \mp \lambda (E^A V^B - E^B V^A). \quad (2.11)$$

It follows that

$$E^A V_A = 0, \quad \mathcal{D}V^A = dV^A + \omega^{AB} V_B \equiv 0. \quad (2.12)$$

The metric tensor is $g_{\underline{mn}} = E_{\underline{m}}^A E_{\underline{n}}^B \eta_{AB}$. In these notations, the background $(A)dS_d$ geometry is described by the $(A)dS_d$ connection $W^{AB} = (h^a, \omega_0^{ab})$ satisfying the zero-curvature equation (see, e.g., [41] for more detail)

$$R^{AB}(W) \equiv dW^{AB} + W^A{}_C \wedge W^{CB} = 0. \quad (2.13)$$

The action of the covariant derivative D_0 on an arbitrary $(A)dS_d$ tensor is given by

$$D_0 A^{A_1(s_1), \dots, A_k(s_k)} = dA^{A_1(s_1), \dots, A_k(s_k)} + s_1 W^{A_1}{}_C \wedge A^{CA_1(s_1-1), \dots, A_k(s_k)} + \dots + s_k W^{A_k}{}_C \wedge A^{A_1(s_1), \dots, CA_k(s_k-1)}. \quad (2.14)$$

2.3 Mixed-symmetry bosonic massless fields

Relativistic fields in AdS_d which admit quantum-mechanically consistent formulation are classified according to lowest weight unitary representations of $o(d-1, 2)$.

where p is the height of the upper rectangular block of the Young tableau $Y(s_1, s_2, \dots, s_q)$, *i.e.*

$$s = s_1 = s_2 = \dots = s_p > s_{p+1} \geq \dots \geq s_q > 0, \quad 0 < p \leq q. \quad (2.18)$$

The field $\Phi^{(s_1, \dots, s_q)}(x)$ generalizes the fluctuational part of the metric field in gravitation and will be referred to as *metric-type* field. (One can use either tangent or world indices since they can be converted into each other by the background frame 1-form h^a .)

The trace conditions (2.2), (2.3) imposed on the metric-type field $\Phi^{(s_1, \dots, s_q)}(x)$ generalize the Fronsdal double-tracelessness condition for totally symmetric fields [3] to higher spin fields of any symmetry type. Note that there are other generalizations of Fronsdal trace conditions for mixed symmetry fields in the literature [11, 12, 13]. Our choice differs from some of them in that the double-tracelessness condition is imposed on the upper rectangular block while the rest of the Young tableau obeys the single-tracelessness condition (the distinguished role of the upper rectangular block is clear from (2.16) and will also be commented on later). The discrepancy between our approach and others comes to light for non-rectangular Young tableaux which contain at least two different blocks of length 2 or more. The simplest example is provided by the $Y(3, 2)$ tableau.

In our formulation, the flat space higher spin gauge transformations have a structure analogous to that proposed in [12]

$$\delta \Phi^{(s_1, \dots, s_q)} = \sum_{i=p}^q \mathcal{P}^{(i)} (\mathcal{D} \xi_i^{(s_1, \dots, s_{i-1}, \dots, s_q)}) , \quad (2.19)$$

where the gauge parameters $\xi_i^{(s_1, \dots, s_{i-1}, \dots, s_q)}$ are described by various Young tableaux $Y(s_1, \dots, s_{i-1}, \dots, s_q)$ provided that $s_i - 1 \geq s_{i+1}$. The gauge parameter $\xi_p^{(s_1, \dots, s_{p-1}, s_{p+1}, \dots, s_q)}$ belongs to $B_{p-1}^{d-1,1}(s_1, \dots, s_{p-1}, s_{p+1}, \dots, s_q)$, while $\xi_k^{(s_1, \dots, s_{k-1}, \dots, s_q)}$ for $k > p$ belongs to $B_p^{d-1,1}(s_1, \dots, s_{k-1}, \dots, s_q)$. $\mathcal{P}^{(i)}$ in (2.19) are projectors that involve appropriate Young symmetrizations and take proper account of traces to project the r.h.s. to $B_p^{d-1,1}(s_1, \dots, s_q)$.

From the unitarity requirement it follows [22] that gauge symmetries for a mixed-symmetry type field are different in flat and AdS_d backgrounds. According to [22] the higher spin gauge symmetries in (2.19) with the parameters $\xi_i^{(s_1, \dots, s_{i-1}, \dots, s_q)}$ with $i > p$ are absent in the AdS_d background. In other words, to obtain the correct set of AdS_d gauge parameters one is allowed to cut a cell from the upper rectangular block only. Parameters with $i > p$ appear in the flat limit. They can play a role in the AdS_d theory as Stueckelberg symmetry parameters, however [23].

2.4 Frame-like formulation for mixed-symmetry bosonic massless fields

The idea of our approach is to replace the *metric-type* field $\Phi^{(s_1, \dots, s_q)}(x)$ (2.17) by the *frame-type* p -form field

$$\begin{aligned} \omega_{(p)}^{a_1(s-1), \dots, a_p(s-1), a_{p+1}(s_{p+1}), \dots, a_q(s_q)} \\ = dx^{\underline{n}_1} \wedge \dots \wedge dx^{\underline{n}_p} \omega_{[\underline{n}_1 \dots \underline{n}_p]}^{a_1(s-1), \dots, a_p(s-1), a_{p+1}(s_{p+1}), \dots, a_q(s_q)}, \end{aligned} \quad (2.20)$$

which takes values in the traceless tensor representation $B_0^{d-1,1}(s-1, \dots, s-1, s_{p+1}, \dots, s_q)$ of the Lorentz group. In other words, we cut off the last column in the upper rectangular block of $Y(s_1, \dots, s_q)$ replacing it by independent p -form indices. The gauge symmetries of the p -form field $\omega_{(p)}$ are required to be of the form

$$\delta \omega_{(p)}^{(s-1, \dots, s_q)} = \mathcal{D} \xi_{(p-1)}^{(s-1, \dots, s_q)} + \sum_{l=p+1}^{q+1} \mathcal{P}^{(i)} (h \wedge \xi_{(p-1)}^{(s-1, \dots, s_l+1, \dots, s_q, 0)}). \quad (2.21)$$

The $(p-1)$ -form gauge parameter $\xi_{(p-1)}^{(s-1, \dots, s_q)}$, which generalizes the linearized diffeomorphism transformation of the frame field, belongs to the same representation $B_0^{d-1,1}(s-1, \dots, s-1, s_{p+1}, \dots, s_q)$ of the tangent Lorentz group as $\omega_{(p)}$. The $(p-1)$ -form shift parameters $\xi_{(p-1)}^{(s-1, \dots, s_l+1, \dots, s_q, s_{q+1})}$, $p+1 \leq l \leq q+1$ (with the convention that $s_{q+1} = 0$) belong to $B_0^{d-1,1}(s-1, \dots, s_l+1, \dots, s_q, s_{q+1})$ and generalize the linearized Lorentz transformations of the frame field. They take values in the tangent Young tableaux which differ from that of $\omega_{(p)}$ by one extra cell in l^{th} row. This extra cell is always contracted with the tangent index of the background frame field h . $\mathcal{P}^{(i)}$ are projectors that take proper account of Young symmetrizations.

The metric-type field is expressed in terms of the component fields of (2.20) as follows

$$\Phi^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)}(x) = \omega^{[a_1 \dots a_p]; a_1(s-1), \dots, a_p(s-1), a_{p+1}(s_{p+1}), \dots, a_q(s_q)}(x), \quad (2.22)$$

i.e. it results from symmetrization of the form indices (converted into the tangent ones) with the tangent indices of first p rows of $\omega_{(p)}$. From this formula it follows that such defined $\Phi^{(s_1, \dots, s_q)}$ belongs to $B_p^{d-1,1}(s_1, \dots, s_q)$. Indeed, the irreducible Young properties are obvious from (2.22) since symmetrization of any index from a lowest row with all indices of some upper row gives zero either because of the Young properties of the tangent indices of the component fields of (2.20) (if a symmetrized index originates from the tangent indices of $\omega_{(p)}$) or because of antisymmetry of the form indices (if a symmetrized index is one of the form indices of $\omega_{(p)}$). Nonzero traces in $\Phi^{(s_1, \dots, s_q)}$ can only result from contractions of the form indices with tangent indices. This just gives the conditions (2.2), (2.3).

The role of the gauge parameters $\xi_{(p-1)}$, which appear in the gauge law (2.21) without derivatives, is to compensate redundant components of the p -form field (2.20) compared to the metric-type field (2.22). The shift symmetry in the gauge law (2.21) compensates all components except for the field $\Phi^{(s_1, \dots, s_q)}$. The simplest

way to see that the shift symmetry parameters do not affect the gauge law for the field $\Phi^{(s_1, \dots, s_q)}$ is to observe that any Lorentz invariant scalar product between $\Phi^{(s_1, \dots, s_q)}$ and shift parameters gives zero as a consequence of their Young properties, *i.e.* they are described by different Young tableaux.

The derivative part of the gauge law (2.21) is such that the gauge transformation of the field $\Phi^{(s_1, \dots, s_q)}$ coincides with (2.19) with all gauge parameters ξ_i with $i > p$ absent. Thus, for the AdS_d background, the p -form field (2.20) with the gauge law (2.21) describes the metric-type field $\Phi^{(s_1, \dots, s_q)}$ with the correct pattern of AdS_d gauge symmetries. It generalizes the 1-form gauge connection $e_{\underline{n}}^{a(s-1)}$ introduced in [6] to describe totally symmetric spin- s gauge fields. Note that the same formalism can be used to describe dynamics in dS_d .

To construct manifestly gauge invariant action one has to introduce more fields which generalize auxiliary and extra fields of [6],[19],[20]. The idea is to associate the shift gauge parameters of (2.21) with some new gauge fields which generalize Lorentz connection. These will be called auxiliary fields while the original p -form (2.20) will be referred to as physical field. The auxiliary fields have transformation laws analogous to (2.21) with the derivative parts containing the shift parameters from the gauge transformation law (2.21) of the physical p -form. In addition, there will be some new shift parameters in the transformation laws of the auxiliary fields. In their turn, these new shift parameters require new gauge fields called extra fields. This procedure extends further to obtain a full set of physical, auxiliary and extra fields necessary to construct curvature $(p+1)$ -forms manifestly invariant under the full set of gauge symmetries. The analysis of the pattern of the full list of additional gauge fields is greatly simplified by the observation applied in [39] to the case of totally symmetric fields that they all result from “dimensional reduction” of a p -form gauge field carrying an appropriate irreducible representation of $o(d-1, 2)$ (or $o(d, 1)$).

2.5 $(A)dS_d$ covariant setup

As explained below, the appropriate set of gauge fields is given by a p -form

$$\Omega_{(p)}^{A_0(r_0), A_1(r_1), \dots, A_q(r_q)} \quad , \quad (2.23)$$

which takes values in the representation of the AdS_d algebra described by the traceless Young tableau $Y(r_0, r_1, \dots, r_q)$ with

$$r_0 = r_1 = \dots = r_p = s - 1, \quad r_i = s_i \quad \text{for} \quad i > p. \quad (2.24)$$

(Contraction of any two tangent indices A with the $o(d-1, 2)$ invariant metric η_{AB} gives zero.) In other words, to describe a massless particle associated with the vacuum energy representation (2.15) $B_0^{d-1,0}(\underbrace{s, \dots, s}_p, s_{p+1}, \dots, s_q)$ of $o(d-1) \subset$

$o(d-1, 2)$ we suggest to use the p -form connection (2.23) which takes values in the representation $B_0^{d-1,2}(\underbrace{s-1, \dots, s-1}_{p+1}, s_{p+1}, \dots, s_q)$ of the AdS_d algebra $o(d-1, 2)$.

The rule therefore is: to obtain the $(A)dS_d$ tensor representation of the gauge field one cuts the shortest column and then adds the longest row to the Young tableau of the vacuum energy representation under consideration. The gauge field is a p -form where p is the height of the cut column of the original vacuum representation.

Tensors from $B_0^{d-1,2}(\underbrace{s-1, \dots, s-1}_{p+1}, s_{p+1}, \dots, s_q)$ can be depicted as

(2.25)

The set of various p -form Lorentz-covariant gauge fields, including physical, auxiliary and extra fields, associated with a particular mixed-symmetry representation of the AdS_d algebra, is in the one-to-one correspondence with the set of irreducible representations of the Lorentz subalgebra $o(d-1, 1) \subset o(d-1, 2)$, contained in $B_0^{d-1,2}(\underbrace{s-1, \dots, s-1}_{p+1}, s_{p+1}, \dots, s_q)$. In practice, Lorentz-covariant component fields

are identified with various independent traceless V -transversal components in the original $o(d-1, 2)$ traceless Young tableau. For the particular case of gravitation it works as in Eq.(2.11). The decomposition of an arbitrary mixed-symmetry field yields the set of p -form Lorentz-covariant tensor fields represented by the following Young tableaux:

(2.26)

where

$$\tilde{s}_{i+1} \leq t_i \leq \tilde{s}_i, \quad (2.27)$$

(with the convention that $\tilde{s}_1 = s - 1$ and $\tilde{s}_{k+1} = 0$). The corresponding Lorentz-covariant tensors result from contractions of some of the indices of the original $o(d - 1, 2)$ tensor with the compensator V^A along with projecting out the V^A transversal components with respect to the rest of indices. In the tableau (2.26) the indices contracted with the compensator are denoted as $\boxed{\blacksquare}$. They disappear from the resulting Lorentz tableau drawn in bold because the compensator is, by definition, Lorentz invariant. Clearly the result is equivalent to the dimensional reduction of an irreducible tensor to one lower dimension⁴. Note that in the list of resulting Lorentz tableaux no two contractions with the compensator hit the same column, i.e., no two cells $\boxed{\blacksquare}$ are situated one under another. This is because the product of two compensators $V^A V^B$ is a symmetric tensor.

⁴The interpretation of the picture (2.26) is somewhat schematic in that respect that the straightforward dimensional reduction of an irreducible Young tableau by contracting some of the indices with the compensator does not generically produce an irreducible lower-dimensional (i.e., Lorentz in our case) Young tableau. Nevertheless, one can see that the list of the resulting irreducible components is correctly reproduced by the bold Young tableaux in (2.26).

To summarize, the decomposition of the $(A)dS_d$ p -form gauge field with tangent indices given by the traceless $(o(d-1,2))o(d,1)$ Young tableau (2.25) into a set of Lorentz-covariant p -form fields is

$$\Omega_{(p)} \rightarrow \bigoplus_{(t_1, \dots, t_k)} \omega_{(p)}^{(t_1, \dots, t_k)}, \quad (2.28)$$

where the fields $\omega_{(p)}^{(t_1, \dots, t_k)}$ parameterized by the integers t_i (2.27) have tangent indices given by various traceless $o(d-1,1)$ Young tableaux (2.26), (2.27).

The dynamical interpretation of different Lorentz-covariant fields is as follows:

- The physical field corresponds to the tableau (2.26) with $t_i = \tilde{s}_{i+1}$ for all i , which is equivalent to cut off the upper row in the Young tableau (2.25). This means that the physical field is identified with the maximally V -tangential component of the $(A)dS$ field (2.23), associated with the contraction of its $s-1$ indices with V^A , *i.e.*

$$\omega_{(p)}^{a_1(s-1), \dots, a_p(s-1), a_{p+1}(s_{p+1}), \dots, a_q(s_q)} = \underbrace{V_{A_0} \dots V_{A_0}}_{s-1} \Omega_{(p)}^{A_0(s-1), a_1(s-1), \dots, a_q(s_q)}. \quad (2.29)$$

Note that contraction of any s indices of $\Omega_{(p)}^{A_0(s-1), \dots, A_q(s_q)}$ with V^A gives zero because of the Young properties of $\Omega_{(p)}^{A_0(s-1), \dots, A_q(s_q)}$.

- The auxiliary fields have tangent Lorentz tableaux which differ from the physical field by one cell, *i.e.* they correspond to tableaux (2.26) with $t_j = \tilde{s}_{j+1} + 1$ for some particular j , while $t_i = \tilde{s}_{i+1}$ for all other i . There are k different auxiliary fields for a Young tableau composed of k blocks (2.25).
- The class of extra fields includes all the rest Lorentz tableaux (2.26) having two or more additional cells compared to the physical field tableau.

The linearized higher spin curvature $(p+1)$ -form associated with the gauge p -form field (2.23) - (2.25) is

$$R_{(p+1)}^{A_0(s-1), \dots, A_q(s_q)} = D_0 \Omega_{(p)}^{A_0(s-1), \dots, A_q(s_q)}, \quad (2.30)$$

where the $(o(d-1,2))o(d,1)$ covariant derivative D_0 is defined according to (2.14) with respect to some background $(A)dS_d$ connection W^{AB} (2.13).

The curvature $(p+1)$ -form is manifestly invariant

$$\delta R_{(p+1)}^{A_0(s-1), \dots, A_q(s_q)} = 0 \quad (2.31)$$

under the gauge transformations

$$\delta \Omega_{(p)}^{A_0(s-1), \dots, A_q(s_q)} = D_0 \xi_{(p-1)}^{A_0(s-1), \dots, A_q(s_q)} \quad (2.32)$$

with the $(p-1)$ -form gauge parameter $\xi_{(p-1)}^{A_0(s-1), \dots, A_q(s_q)}$ and satisfies Bianchi identities

$$D_0 R_{(p+1)}^{A_0(s-1), \dots, A_q(s_q)} = 0 \quad (2.33)$$

as a consequence of the zero-curvature condition $D_0^2 = 0$ (2.13). Another consequence of the zero-curvature condition is that the gauge transformations (2.32) are reducible. There exists the set of level- $(l+2)$ ($0 \leq l \leq p-2$) gauge parameters and gauge transformations of the form

$$\delta \xi_{(p-l-1)}^{A_0(s-1), \dots, A_q(s_q)} = D_0 \xi_{(p-l-2)}^{A_0(s-1), \dots, A_q(s_q)} . \quad (2.34)$$

The gauge transformation law (2.32) gives precise form of (2.21) along with gauge transformations for all auxiliary and extra fields.

2.6 Fermionic mixed-symmetry massless fields

Formulation of mixed-symmetry fermionic massless fields is analogous. Consider a fermionic field which describes upon quantization the unitary module of the AdS_d symmetry group $o(d-1, 2)$ induced from the vacuum module of its maximal compact subgroup $o(2) \oplus o(d-1)$, characterized by some energy E_0 and “spin” $\mathbf{s} = (h_1, \dots, h_q, 1/2, \dots, 1/2)$ with $h_1 \geq h_2 \geq \dots \geq h_q > 1/2$, where all $2h_i$ are odd and $q \leq \nu = [\frac{d-1}{2}]$. In terms of the Young tableau associated with the corresponding spinor-tensor representation of $o(d-1)$, $s_i = (h_i - 1/2)$ is the length of its i^{th} row. The vacuum energy E_0 of massless fermionic fields is [22]

$$E_0 = s_1 - p + d - \frac{3}{2} , \quad (2.35)$$

where p is the height of the upper rectangular block.

Introduce a Lorentz-covariant *spinor-tensor* field

$$\psi^\alpha |a_1(s_1), a_2(s_2), \dots, a_q(s_q) , \quad (2.36)$$

which is symmetric in each group of indices $a_i(s_i)$ and satisfies the Young symmetry conditions associated with the Young tableau $Y(s_1, s_2, \dots, s_q)$. (α is Lorentz spinor index.)

Let the spinor-tensor (2.36) satisfy the conditions

$$(\gamma_{a_i} \gamma_{a_i} \gamma_{a_i})^\alpha{}_\beta \psi^\beta |a_1(s_1), a_2(s_2), \dots, a_q(s_q) = 0 , \quad 0 < i \leq m , \quad (2.37)$$

and

$$\gamma_{a_i}{}^\alpha{}_\beta \psi^\beta |a_1(s_1), a_2(s_2), \dots, a_q(s_q) = 0 , \quad m < i \leq q , \quad (2.38)$$

where γ_a are Dirac matrices, $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, and m is some non-negative integer. The linear space of $o(d-1, 1)$ spinor-tensors (2.36) which have the Young properties of the type $Y(s_1, \dots, s_q)$ and satisfy the conditions (2.37), (2.38) will be denoted $F_m^{d-1,1}(s_1, \dots, s_q)$ (respectively, $F_m^{p,r}(s_1, \dots, s_q)$ for $o(p, r)$). It follows from the condition (2.37) that contraction of any two pairs of indices from any of the first m rows of the Young tableau gives zero. Also, it follows from (2.38) that contraction of any pair of indices from the last $q-m$ rows gives zero.

To describe AdS_d dynamics of a spin- \mathbf{s} massless fermion, introduce a Lorentz-covariant *spinor-tensor* field

$$\psi^\alpha |a_1(s_1), a_2(s_2), \dots, a_q(s_q)(x) \in F_p^{d-1,1}(s_1, \dots, s_q) , \quad (2.39)$$

where p is the height of the upper rectangular block of the Young tableau $Y(s_1, s_2, \dots, s_q)$, *i.e.*

$$s = s_1 = s_2 = \dots = s_p > s_{p+1} \geq \dots \geq s_q > 0, \quad 0 < p \leq q. \quad (2.40)$$

Analogously to the case of bosonic fields, the metric-type field (2.39) is replaced by the p -form spinor-tensor field⁵

$$\Omega_{(p)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q), \quad (2.41)$$

which take values in the representation $F_0^{d-1,2}(\underbrace{s-1, \dots, s-1}_{p+1}, s_{p+1}, \dots, s_q)$ of $o(d-1, 2)$. Here $\hat{\alpha}$ is some irreducible $o(d-1, 2)$ spinor index, *i.e.* it is Majorana or Weyl or both, whenever possible. The Lorentz spinor index α is then identified with the $o(d-1, 2)$ spinor index $\hat{\alpha}$, which means that it is not necessarily irreducible (e.g., chiral) with respect to $o(d-1, 1)$.

The background covariant derivative acts on an arbitrary AdS_d spinor-tensor in the standard way

$$\begin{aligned} D_0 \Upsilon^{\hat{\alpha}} |A_1(s_1), \dots, A_m(s_m) \\ = d \Upsilon^{\hat{\alpha}} |A_1(s_1), \dots, A_m(s_m) + \frac{1}{2} W_{BC} (\sigma^{BC})^{\hat{\alpha}}_{\hat{\beta}} \wedge \Upsilon^{\hat{\beta}} |A_1(s_1), \dots, A_m(s_m) \\ + s_1 W^{A_1}_C \wedge \Upsilon^{\hat{\alpha}} |CA_1(s_1-1), \dots, A_m(s_m) + \dots + s_m W^{A_m}_C \wedge \Upsilon^{\hat{\alpha}} |A_1(s_1), \dots, CA_m(s_m-1), \end{aligned} \quad (2.42)$$

where the background connection W^{AB} satisfies the zero-curvature condition (2.13) and

$$\sigma^{AB} = \frac{1}{4} [\gamma^A, \gamma^B], \quad \{\gamma_A, \gamma_B\} = 2\eta_{AB}.$$

The fermionic curvature

$$R_{(p+1)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q) = D_0 \Omega_{(p)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q) \quad (2.43)$$

is invariant

$$\delta R_{(p+1)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q) = 0 \quad (2.44)$$

under the gauge transformations

$$\delta \Omega_{(p)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q) = D_0 \xi_{(p-1)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q). \quad (2.45)$$

The physical field is

$$\omega_{(p)}^{\alpha} |a_1(s-1), \dots, a_p(s-1), a_{p+1}(s_{p+1}), \dots, a_q(s_q) = \underbrace{V_{A_0} \dots V_{A_0}}_{s-1} \Omega_{(p)}^{\hat{\alpha}} |A_0(s-1), a_1(s-1), \dots, a_q(s_q). \quad (2.46)$$

⁵The analogous construction was exploited in [70, 40] to describe symmetric massless fermions in AdS_5 .

In the fermionic case all other components in

$$\Omega_{(p)}^{\hat{\alpha}} |A_0(s-1), \dots, A_q(s_q) \quad (2.47)$$

will be called extra fields. The metric-type component is defined in terms of physical field analogously to the bosonic case (2.22)

$$\psi^{\alpha} |a_1(s_1), a_2(s_2), \dots, a_q(s_q)(x) = \omega^{\alpha} |a_1 \dots a_p; a_1(s-1), \dots, a_p(s-1), a_{p+1}(s_{p+1}), \dots, a_q(s_q)(x). \quad (2.48)$$

As a result, it satisfies the conditions (2.37) and (2.38) and belongs to $F_p^{d-1,1}(s_1, \dots, s_q)$.

2.7 Action and Weyl tensors

Having constructed the gauge invariant linearized curvatures (2.30) and (2.43) one can look for a free action functional in the form [20]

$$\mathcal{S}_2 = \int_{\mathcal{M}^d} \alpha^{\dots}(V) \underbrace{E^{\dots} \wedge \dots \wedge E^{\dots}}_{d-2p-2} \wedge R_{(p+1)}^{\dots} \wedge R_{(p+1)}^{\dots}. \quad (2.49)$$

Here E is the background frame field and $\alpha^{\dots}(V)$ are some coefficients which parameterize various types of index contractions between curvatures, frame fields and compensators. Any such action is gauge invariant with respect to the full set of gauge transformations because of (2.31), (2.32) and (2.44), (2.45).

The coefficients have to be determined by imposing *the extra field decoupling condition* which, effectively, requires the action to be free of higher derivatives of the physical field. Actually, in the fermionic case, the higher spin equations should be first-order that will be true if the variation of \mathcal{S}_2 with respect to all extra fields is demanded to identically vanish [21]. In the bosonic case, the higher spin equations are of second-order. As the auxiliary fields are expressed by virtue of their equations of motion in terms of first derivatives of the metric-type field $\Phi^{(s_1, \dots, s_q)}$ modulo pure gauge parts, the bosonic equations of motion will be of second-order once the variation of \mathcal{S}_2 with respect to extra fields vanishes identically [20]. As soon as the coefficients that guarantee independence of extra fields are found, the resulting action will, by construction, be invariant under correct higher spin gauge transformations and give rise to invariant differential equations for the metric-type field $\Phi^{a_1(s_1), a_2(s_2), \dots, a_q(s_q)}(x)$ in the bosonic case and $\psi^{\alpha} |a_1(s_1), a_2(s_2), \dots, a_q(s_q)(x)$ in the fermionic case. Realization of this program for a general massless field will be given elsewhere. In Section 3 we illustrate how it works for some simple examples.

Although extra fields do not contribute to the free higher spin action, they play a role at the interaction level. It is therefore necessary to express auxiliary and extra fields in terms of derivatives of the physical fields by appropriate constraints. The strategy that proved to be most appropriate for the case of symmetric fields [19, 20, 21] is to impose constraints in terms of linearized higher spin curvatures setting to zero as many of their components as possible to express algebraically auxiliary and extra fields in terms of derivatives of the physical fields. These constraints generalize the zero-torsion constraint in gravity which expresses Lorentz connection in terms of first derivatives of the physical field identified with the frame 1-form.

The analysis of constraints for the case of generic mixed symmetry massless fields is important in several respects. In particular, it provides a starting point towards the unfolded formulation of nonlinear higher spin dynamics in the form of appropriate covariant constancy conditions. For the case of totally symmetric massless fields it is known [20, 21] that the constraints for auxiliary and extra fields along with the field equations and Bianchi identities allow one to set equal to zero most of the components of the linearized higher spin curvatures. By analogy with gravity, the non-zero components are called higher spin Weyl tensors. As imposed constraints express all fields in terms of derivatives of the physical higher spin field, the generalized Weyl tensors turn out to be expressed in terms of derivatives of the physical field and, being components of the higher spin curvatures, remain invariant under the higher spin gauge transformations. As a result, generalized Weyl tensors parameterize those gauge invariant combinations of derivatives of the physical fields which remain nonzero on-mass-shell. The full analysis of their structure, as well as of the structure of constraints, is beyond the scope of this paper. Here we only would like to note that our construction naturally gives rise to the Weyl tensors analogous to the gauge invariant mixed symmetry higher spin curvatures found by Medeiros and Hull in [17] within non-local formulation of higher spin dynamics in Minkowski space.

Since an irreducible AdS_d system decomposes in the flat limit into a set of independent mixed-symmetry Minkowski higher spin fields, it is natural to conjecture that the Weyl tensors associated with any irreducible flat space subsystem should appear. From the analysis of [23, 17] it follows that the set of flat space generalized Weyl tensors associated with the AdS_d mixed-symmetry field, described in terms of a p -form connection taking values in the AdS_d Young tableau (2.25), is given by the set of various Young tableaux of the form

$$(2.50)$$

where

$$\tilde{s}_{i+1} \leq n_i \leq \tilde{s}_i \quad (2.51)$$

(with the convention that $2 \leq i \leq k$ and $\tilde{s}_{k+1} = 0$). In the AdS_d case, Weyl tensors (2.50) are components of some irreducible $o(d-1, 2)$ -module. This implies that Weyl tensors (2.50) should be related to each other by some differential equations which express compatibility conditions (*i.e.*, Bianchi identities) for the expressions of higher spin curvatures in terms of Weyl tensors. Systematic analysis of these relations to be presented elsewhere will lead to the full unfolded formulation of the higher spin dynamics for free mixed fields in AdS_d .

The Weyl tensors (2.50), (2.51) contain the primary Weyl tensor

$$C^{a_0(s), \dots, a_p(s), a_{p+2}(\tilde{s}_2), \dots, a_{p+p_2}(\tilde{s}_2), \dots, a_q(\tilde{s}_k)} \quad (2.52)$$

associated with the Young tableau (2.50) having the minimal possible number of cells

(2.53)

It can be interpreted as the invariant tensor of [17] associated with the flat space mixed-symmetry field described by the Young tableau with the minimal number of cells. In the AdS_d background all other Weyl tensors in (2.50) turn out to be expressed through derivatives of the primary Weyl tensor (2.53) by virtue of Bianchi identities.

The primary Weyl tensor (2.52) parameterizes the components of the curvature

$$R_{(p+1)}^{a_0(s-1), \dots, a_p(s-1), a_{p+2}(\tilde{s}_2), \dots, a_{p+p_2}(\tilde{s}_2), \dots, a_q(\tilde{s}_k)}(x) \quad (2.54)$$

defined by the formula

$$\begin{aligned} & C^{a_0(s), \dots, a_p(s), a_{p+2}(\tilde{s}_2), \dots, a_{p+p_2}(\tilde{s}_2), \dots, a_q(\tilde{s}_k)}(x) \\ &= R^{[a_0 \dots a_p]; a_0(s-1), \dots, a_p(s-1), a_{p+2}(\tilde{s}_2), \dots, a_{p+p_2}(\tilde{s}_2), \dots, a_q(\tilde{s}_k)}(x). \end{aligned} \quad (2.55)$$

As follows from the analysis of [20, 38, 39], primary Weyl tensors are classified by the cohomology group $H^{p+1}(\sigma_-)$, where σ_- is the part of the full AdS_d covariant derivative that decreases a number of Lorentz indices (σ_- was called τ_- in [20]). It has the following structure

$$\sigma_-(R)^{a_0(r_0), \dots, a_q(r_q)} = \sum_i \alpha_i(r_j) \mathcal{P}^{(i)} \left(E_{a_i} \wedge R^{a_0(r_0), \dots, a_i(r_i+1), \dots, a_q(r_q)} \right), \quad (2.56)$$

where $\alpha_i(r_j)$ are some coefficients and $\mathcal{P}^{(i)}$ are projectors that guarantee that l.h.s. of (2.56) is some Young tableau $Y(r_0, \dots, r_q)$ from the list of Lorentz representations (2.26), (2.27) associated with the system under consideration⁶. As a consequence of the flatness of the AdS_d covariant derivative (2.13), the operator σ_- turns out to be nilpotent, $\sigma_-^2 = 0$. (Note also that the operator σ_- is Lorentz invariant.) Then one can see that Bianchi identities require a part of the curvature associated with a primary Weyl tensor to be σ_- closed, *i.e.* to satisfy $\sigma_-(R) = 0$. On the other hand any σ_- exact part of R can be adjusted to zero by an appropriate choice of constraints for auxiliary and extra fields.

It is elementary to see that the curvature (2.55) belongs to $H^{p+1}(\sigma_-)$ using a basis for Young tableaux with antisymmetries associated with the columns manifest. Indeed, the application of σ_- (2.56) may give a non-zero result only if the index of the frame field is contracted with some of the indices of the shortest columns of height $p+1$ (all other terms are projected to zero by the projectors in (2.56) because any Young tableaux with a cell cut from any other column is not in the list of the Lorentz representations associated with the chosen field). In that case, the result is also zero because of the Young property of Young tableaux in the antisymmetric basis, which requires that antisymmetrization of all indices of some column and any index from some other column of less or equal height gives zero. The antisymmetrization over $p+2$ indices results from contraction with $p+2$ frame 1-forms.

Thus the curvature (2.54), (2.55) is σ_- closed. But it cannot be σ_- exact because the Young tableau (2.53) just does not appear among the Lorentz representations contained in exact curvatures. Indeed, tensoring the form indices with the tangent ones for any allowed W in the exact representation $R^{a_0(r_0), \dots, a_q(r_q)} = \sigma_-(W)^{a_0(r_0), \dots, a_q(r_q)}$ it is possible to obtain tableaux with at most p cells in the last right column.

Note that traces contained in the primary Weyl tensor need separate consideration because traces of components having different Young symmetry types may be related. We therefore consider traceless $C^{a_0(s), \dots, a_p(s), a_{p+2}(\tilde{s}_2), \dots, a_{p+p_2}(\tilde{s}_2), \dots, a_q(\tilde{s}_k)}(x) \in B_0^{d-1,1}(\underbrace{s, \dots, s}_{p+1}, \underbrace{\tilde{s}_2, \dots, \tilde{s}_2}_{p_2-1}, \dots, \tilde{s}_k)$ only.

Thus, the frame-like formulation of the mixed symmetry massless higher spin fields we propose leads to the on-mass-shell nontrivial primary Weyl tensor $C^{a_0(s), \dots, a_p(s), a_{p+2}(\tilde{s}_2), \dots, a_{p+p_2}(\tilde{s}_2), \dots, a_q(\tilde{s}_k)}(x)$ invariant under AdS_d higher spin gauge symmetries. By construction, it contains $s - \tilde{s}_2$ derivatives of the physical field⁷ and is described by the Young tableau (2.53). Other tensors in (2.50) contain more (but no more than s) derivatives of the physical field. They are expressed via derivatives of the primary AdS_d Weyl tensor but are expected to become in the flat limit primary Weyl tensors associated with independent flat higher spin subsystems.

⁶For particular examples of mixed-symmetry fields having Young symmetries of the types $Y(2, 1)$ and $Y(2, 2)$ the form of σ_- can be easily read off from Eqs.(3.6), (3.30) of next section.

⁷This is because additional indices in the curvature compared to the original physical field are carried by the space-time derivative operators which appear either through derivatives in the higher spin curvature or upon resolving constraints for auxiliary and extra fields in terms of derivatives of the physical fields.

3 Examples

To illustrate the general scheme described in section 2, we first consider two simplest examples of mixed-symmetry fields, namely, a three-cell “hook” field $\Phi^{a(2),b}(x)$ and a four-cell “window” field $\Phi^{a(2),b(2)}(x)$. In each case we present the full set of p -form gauge fields which consists of the physical and auxiliary fields and build the actions which properly describe irreducible $(A)dS_d$ dynamics⁸. We derive the second-order equations of motion on the metric-type fields $\Phi^{a(2),b}(x)$ and $\Phi^{a(2),b(2)}(x)$ which, in an appropriate gauge reproduce the equations obtained by Metsaev [22]. The sets of fields we use in this section are equivalent to those used by Zinoviev in [30] to describe “hook” and “window” tableaux within first-order formalism. The universal p -form description suggested in this paper makes higher spin symmetries manifest, however. In the flat limit, the “hook” theory yields an additional symmetry not placed in the initial $(A)dS_d$ formulation. The case of “window” tableau is generalized to rectangular two-row Young tableau of an arbitrary length in subsection 3.3. The action is uniquely fixed by *the extra field decoupling condition*.

3.1 Three-cell “hook” tableau

Consider Lorentz-covariant “hook” field $\Phi^{[ab],c}(x)$ which is antisymmetric in the first two indices⁹ $\Phi^{[ab],c}(x) = -\Phi^{[ba],c}(x)$ and satisfies the Young symmetry condition

$$\Phi^{[ab,c]}(x) = 0. \quad (3.1)$$

The Lagrangian formulation for the metric-type field $\Phi^{[ab],c}(x)$ in the flat background was elaborated in [7, 8]. The corresponding action is invariant under the gauge transformation

$$\delta\Phi^{[ab],c} = \partial^a S^{bc} - \partial^b S^{ac} + 2\partial^c \Lambda^{[ab]} - \partial^a \Lambda^{[bc]} + \partial^b \Lambda^{[ac]} \quad (3.2)$$

with antisymmetric gauge parameter $\Lambda^{[ab]}(x) = -\Lambda^{[ba]}(x)$ and symmetric gauge parameter $S^{ab}(x) = S^{ba}(x)$. There is the level-2 gauge transformation with the gauge parameter $\xi^a(x)$

$$\delta S^{ab} = 3(\partial^a \xi^b + \partial^b \xi^a), \quad \delta \Lambda^{[ab]} = \partial^b \xi^a - \partial^a \xi^b. \quad (3.3)$$

The generalization of the flat theory to AdS_d was constructed in [23]. It was shown that an appropriate deformation gives rise to an action invariant under

$$\delta\Phi^{[ab],c} = 2\mathcal{D}^c \Lambda^{[ab]} - \mathcal{D}^a \Lambda^{[bc]} + \mathcal{D}^b \Lambda^{[ac]}, \quad (3.4)$$

while the gauge symmetry with the parameter S^{ab} is lost in AdS_d . The absence of this gauge invariance is in agreement with the fact that physical degrees of freedom

⁸In the case of dS_d space-time we do not require the theory to describe unitary dynamics with bounded energy.

⁹In what follows, we adopt for “hook” and “window” Young tableaux antisymmetric basis notations. Namely, indices placed in square brackets are assumed to be antisymmetrized as $X^{[a}Y^{b]} = \frac{1}{2!}(X^a Y^b - X^b Y^a)$.

of massless AdS_d fields are not described by irreps of Wigner little group $o(d-2)$ [23]. For this particular example, an AdS_d massless “hook” field decomposes in the flat limit into a massless “hook” field and a massless spin-2 symmetric field.

To reformulate the AdS_d theory of the metric-type field $\Phi^{[ab],c}(x)$ [23] within the scheme of section 2 we introduce the physical and auxiliary 1-forms

$$e_{(1)}^{[ab]} = dx^{\underline{n}} e_{\underline{n}}^{[ab]}, \quad \omega_{(1)}^{[abc]} = dx^{\underline{n}} \omega_{\underline{n}}^{[abc]} \quad (3.5)$$

with antisymmetric tangent Lorentz indices. Linearized curvature 2-forms associated with (3.5) are

$$r_{(2)}^{[ab]} = \mathcal{D}e_{(1)}^{[ab]} + h_c \wedge \omega_{(1)}^{[abc]}, \quad \mathcal{R}_{(2)}^{[abc]} = \mathcal{D}\omega_{(1)}^{[abc]} - 3\lambda^2 h^{[a} \wedge e_{(1)}^{bc]}, \quad (3.6)$$

where \mathcal{D} is a background Lorentz-covariant derivative. These curvatures are invariant under the gauge transformations

$$\delta e_{(1)}^{[ab]} = \mathcal{D}\xi_{(0)}^{[ab]} + h_c \xi_{(0)}^{[abc]}, \quad \delta \omega_{(1)}^{[abc]} = \mathcal{D}\xi_{(0)}^{[abc]} - 3\lambda^2 h^{[a} \xi_{(0)}^{bc]} \quad (3.7)$$

with 0-form gauge parameters $\xi_{(0)}^{[ab]}$ and $\xi_{(0)}^{[abc]}$ antisymmetric in tangent indices. The gauge symmetry implies the following Bianchi identities

$$\mathcal{D}r_{(2)}^{[ab]} + h_c \wedge \mathcal{R}_{(2)}^{[abc]} = 0, \quad \mathcal{D}\mathcal{R}_{(2)}^{[abc]} - 3\lambda^2 h^{[a} \wedge r_{(2)}^{bc]} = 0. \quad (3.8)$$

To see how the metric-type field $\Phi^{[ab],c}(x)$ is encoded in the gauge field $e_{\underline{n}}^{[ab]}(x)$ with the gauge law (3.7) we decompose the Lorentz-covariant 1-form gauge fields (3.5) into different Young symmetry type components as

$$e_{\underline{n}}^{[ab]} \sim \Phi^{[ab],c} \oplus X^{[abc]}, \quad (3.9)$$

$$\omega_{\underline{n}}^{[abc]} \sim \omega^{[abc],d} \oplus Y^{[abcd]}, \quad (3.10)$$

where the tensors $\Phi^{[ab],c}$ and $\omega^{[abc],d}$ contain their traces. The tensor $\Phi^{[ab],c} \in B_1^{d-1,1}(2,1)$ is identified with the dynamical metric-type field. Its gauge transformation derived from (3.7) reads

$$\delta \Phi^{[ab],c} = 2\mathcal{D}^c \xi_{(0)}^{[ab]} - \mathcal{D}^a \xi_{(0)}^{[bc]} + \mathcal{D}^b \xi_{(0)}^{[ac]} \quad (3.11)$$

and is in agreement with (3.4). The totally antisymmetric component $X^{[abc]}$ of $e_{\underline{n}}^{[ab]}(x)$ is compensated by the gauge shift generated by the 0-form gauge parameter $\xi_{(0)}^{[abc]}$ (3.7).

The Lorentz-covariant fields combine into a single 1-form field with $(A)dS_d$ tangent indices as

$$\Omega_{(1)}^{[ABC]} = e_{(1)}^{[ab]} \oplus \omega_{(1)}^{[abc]}, \quad R_{(2)}^{[ABC]} = r_{(2)}^{[ab]} \oplus \mathcal{R}_{(2)}^{[abc]}, \quad \xi_{(0)}^{[ABC]} = \xi_{(0)}^{[ab]} \oplus \xi_{(0)}^{[abc]}. \quad (3.12)$$

The gauge transformations, curvature and Bianchi identities take now the form

$$\delta \Omega_{(1)}^{[ABC]} = D_0 \xi_{(0)}^{[ABC]}, \quad R_{(2)}^{[ABC]} = D_0 \Omega_{(1)}^{[ABC]}, \quad D_0 R_{(2)}^{[ABC]} = 0, \quad (3.13)$$

where D_0 is the background $(A)dS_d$ derivative (2.13), (2.14).

The most general parity-invariant action is written in terms of $(A)dS_d$ covariant tensors

$$\begin{aligned} \mathcal{S}_2 = & \frac{\kappa_1}{\lambda^2} \int_{\mathcal{M}^d} \epsilon_{ABCD M_5 \dots M_{d+1}} E^{M_5} \wedge \dots \wedge E^{M_d} V^{M_{d+1}} \wedge R_{(2)}^{ABE} \wedge R_{(2)}^{CD}{}_E \\ & + \frac{\kappa_2}{\lambda^2} \int_{\mathcal{M}^d} \epsilon_{ABCD M_5 \dots M_{d+1}} E^{M_5} \wedge \dots \wedge E^{M_d} V^{M_{d+1}} \wedge R_{(2)}^{ABE} \wedge R_{(2)}^{CDF} V_E V_F . \end{aligned} \quad (3.14)$$

Here $\kappa_{1,2}$ are arbitrary dimensionless constants. By adding the total derivative term

$$\frac{1}{\lambda^2} \int_{\mathcal{M}^d} d \left(\epsilon_{ABCDE M_6 \dots M_{d+1}} E^{M_6} \wedge \dots \wedge E^{M_d} V^{M_{d+1}} \wedge R_{(2)}^{ABC} \wedge R_{(2)}^{DE}{}_F V^F \right) , \quad (3.15)$$

the freedom in $\kappa_{1,2}$ can be fixed up to an overall multiplicative factor in front of the action (3.14). The variation of the action yields the following equations of motion:

$$\frac{\delta \mathcal{S}_2}{\delta \Omega_{(1)}^{[ABC]}} = 0 \quad \Leftrightarrow \quad \epsilon^{[AB}{}_{M_1 \dots M_{d-1}} E^{M_3} \wedge \dots \wedge E^{M_{d-1}} \wedge R_{(2)}^{C]M_1 M_2} = 0 . \quad (3.16)$$

Converting all world indices into tangent ones, the equation of motion (3.16) can be rewritten in terms of Lorentz-covariant components (3.12) as

$$\frac{\delta \mathcal{S}_2}{\delta e_{(1)}^{[ab]}} = 0 \quad \Rightarrow \quad (\mathcal{R}_{st}^{[b; st} \delta^a]_d + 2 \mathcal{R}_{dt}^{[b; ta]}) = 0 , \quad (3.17)$$

$$\frac{\delta \mathcal{S}_2}{\delta \omega_{(1)}^{[abc]}} = 0 \quad \Rightarrow \quad (r_k^{[c; bk} \delta^a]_d + r_d^{[c; ab]}) = 0 . \quad (3.18)$$

The general solution for these linear restrictions on the curvatures r and \mathcal{R} which conform Bianchi identities (3.8) is

$$r_{(2)}^{[ab]} = h_c \wedge h_d T^{[ab], [cd]} , \quad (3.19)$$

$$\mathcal{R}_{(2)}^{[abc]} = h_d \wedge h_f C^{[abc], [df]} , \quad (3.20)$$

where 0-forms $C^{[abc], [df]}$ and $T^{[ab], [cd]}$ are described by traceless two-column Young tableaux

$$C^{[abc], [df]} \eta_{cd} = 0 , \quad \lambda^2 T^{[ab], [cd]} \eta_{bc} = 0 . \quad (3.21)$$

They parameterize those components of the field strengths which can be nonzero on-mass-shell and are called higher spin Weyl tensors. The tensor $T^{[ab], [cd]}$ is the particular case of the primary Weyl tensor (2.53). The tensor $C^{[abc], [df]}$, corresponds to the additional Weyl tensors (2.50) and can be expressed as a first derivative of $T^{[ab], [cd]}$. Note that in the limit $\lambda = 0$ the tracelessness condition for $T^{[ab], [cd]}$ disappears. This reflects the fact of appearance of an additional symmetry in Minkowski space (see below).

The equation (3.18) is the constraint which expresses the auxiliary field in terms of derivatives of the physical field. By gauge fixing $X^{[abc]}$ to zero (cf. (3.7), (3.9)) one finds

$$\omega^{[abc],d} = -\frac{1}{2}(\mathcal{D}^a\Phi^{[bc],d} - \mathcal{D}^b\Phi^{[ac],d} + \mathcal{D}^c\Phi^{[ab],d}). \quad (3.22)$$

For this gauge it follows that $Y^{abcd} = 0$ in (3.10). Substituting (3.22) into the equation (3.17) which contains first derivatives of the auxiliary field, one finds the second-order equation on the metric-type field $\Phi^{[ab],c}$

$$(\mathcal{D}^2 + \dots + 3\lambda^2)\Phi^{[ab],c} = 0, \quad (3.23)$$

where terms containing $\mathcal{D}_c\Phi^{[ca],b}$, $\mathcal{D}_c\Phi^{[ab],c}$ or $\Phi^{[ab],b}$ are omitted. The covariant D'Alembertian is

$$\mathcal{D}^2 \equiv \mathcal{D}_a\mathcal{D}^a = h^{\underline{n}}_a\mathcal{D}_{\underline{n}}(h^{\underline{m}a}\mathcal{D}_{\underline{m}}), \quad (3.24)$$

where $\mathcal{D}_{\underline{m}}$ is the background Lorentz-covariant derivative (2.10). This equation coincides with that found in [23] and, in the covariant gauge $\Phi^{[ab],b} = 0$, reproduces the equation found by Metsaev [22].

By virtue of Bianchi identities (3.8), the second-order equation (3.23) can be equivalently rewritten in the form

$$\eta_{ad}\mathcal{D}^{[a}F^{bc],df} = 0, \quad (3.25)$$

where $F^{[ab],[cd]} = \mathcal{D}^{[a}\Phi^{cd],b} + \mathcal{D}^{[c}\Phi^{ab],d}$ is the $Y(2,2)$ projection of the physical curvature $r^{[ab]}_{(2)}$ (3.6). By construction, the tensor $F^{[ab],[cd]}$ is invariant under the gauge transformations of the metric-like field (3.11). This implies that the second-order equation of motion (3.25) which has the symmetries of the field $\Phi^{[ab],c}$ is gauge invariant.

Now let us discuss the flat limit of the action (3.14) and equations (3.23), (3.25). As is seen from the equations of motion (3.16), all terms containing poles $1/\lambda$ enter the action through total derivatives. As a result, the action admits a well defined flat limit at $\lambda \rightarrow 0$ (3.14) upon adjusting appropriate total derivative terms carrying negative powers of λ . Indeed, making use of the freedom in the parameters $\kappa_{1,2}$ (3.14), the action (3.14) can be rewritten in the form valid both for the flat and $(A)dS_d$ backgrounds

$$\mathcal{S}_2 = \int_{\mathcal{M}^d} \epsilon_{abcdm_5\dots m_d} h^{m_5} \wedge \dots \wedge h^{m_d} \wedge r^{ab}_{(2)} \wedge r^{cd}_{(2)}. \quad (3.26)$$

For the flat space case one replaces $\mathcal{D}_{\underline{m}} \rightarrow \partial_{\underline{m}}$. The equations of motion have the form (3.19) and (3.20) at $\lambda = 0$. However, a special feature of the theory in the flat limit is that, in accordance with (3.2), an additional symmetry with the symmetric parameter $S^{ab} = S^{ba}$, $S^a_a \neq 0$ appears in flat background [23] for the second-order field equations (3.23). Note that trivialization of the second equation in (3.21) in the flat limit is just the Noether identity for this new symmetry.

It is worth to comment that in our formalism there is a systematic way to show that the gauge invariance with the symmetric parameter S^{ab} appears in the flat limit

by observing that the expression (3.22) for the auxiliary field $\omega^{[abs],d}(\Phi)$ in terms of the metric-type field turns out to be invariant under the gauge transformation with the parameter S^{ab} in the flat limit. The existence of two types of first-order invariant expressions $F^{[ab],[cd]}(\Phi)$ and $\omega^{[abc],d}(\Phi)$ for the “hook” field which are invariant under gauge transformations with antisymmetric and symmetric parameters, respectively, was originally found in [7, 29]. These tensors get natural geometric interpretation of particular field strength and connection in our geometric approach. The difference in their interpretation is because only one of the two types of symmetries remains unbroken in AdS_d .

3.2 Four-cell “window” tableau

The mixed-symmetry field described by the four-cell rectangular “window” tableau was considered in [13, 64, 65] for the flat background and in [29] for the $(A)dS_d$ background. Here we demonstrate how our approach reproduces the analysis of [13, 29].

Consider Lorentz-covariant “window” field $\Phi^{[ab],[cd]}(x)$ antisymmetric in the first and second groups of indices and satisfying the Young symmetry condition

$$\Phi^{[ab,c]d}(x) = 0. \quad (3.27)$$

The corresponding gauge symmetry is given by

$$\delta\Phi^{[ab],[cd]} = \mathcal{D}^a S^{[cd],b} - \mathcal{D}^b S^{[cd],a} + \mathcal{D}^c S^{[ab],d} - \mathcal{D}^d S^{[ab],c} \quad (3.28)$$

with the parameter $S^{[ab],c}(x)$ antisymmetric in the first two indices and satisfying the Young symmetry condition $S^{[ab,c]}(x) = 0$. The gauge parameter can be chosen either traceless [13] or not [29]. In the latter case the gauge transformations (3.28) are reducible: the transformation $\delta S^{[ab],c} = 2\mathcal{D}^c A^{ab} - \mathcal{D}^a A^{bc} + \mathcal{D}^b A^{ac}$ with antisymmetric parameter $A^{[ab]}$ leaves invariant $\delta\Phi^{[ab],[cd]}$ (3.28).

In accordance with the general prescription of section 2 introduce the physical and auxiliary 2-form fields

$$e_{(2)}^{[ab]} = dx^{\underline{m}} \wedge dx^{\underline{n}} e_{[\underline{mn}]}^{[ab]}, \quad \omega_{(2)}^{[abc]} = dx^{\underline{m}} \wedge dx^{\underline{n}} \omega_{[\underline{mn}]}^{[abc]} \quad (3.29)$$

with antisymmetric tangent Lorentz indices. Linearized curvature 3-forms are

$$r_{(3)}^{[ab]} = \mathcal{D}e_{(2)}^{[ab]} + h_c \wedge \omega_{(2)}^{[abc]}, \quad \mathcal{R}_{(3)}^{[abc]} = \mathcal{D}\omega_{(2)}^{[abc]} - 3\lambda^2 h^{[a} \wedge e_{(2)}^{bc]}. \quad (3.30)$$

They satisfy Bianchi identities

$$\mathcal{D}r_{(3)}^{[ab]} + h_c \wedge \mathcal{R}_{(3)}^{[abc]} = 0, \quad \mathcal{D}\mathcal{R}_{(3)}^{[abc]} - 3\lambda^2 h^{[a} \wedge r_{(3)}^{bc]} = 0 \quad (3.31)$$

and are invariant under the gauge transformations

$$\delta e_{(2)}^{[ab]} = \mathcal{D}\xi_{(1)}^{[ab]} + h_c \wedge \xi_{(1)}^{[abc]}, \quad \delta\omega_{(2)}^{[abc]} = \mathcal{D}\xi_{(1)}^{[abc]} - 3\lambda^2 h^{[a} \wedge \xi_{(1)}^{bc]} \quad (3.32)$$

with the 1-form gauge parameters $\xi_{(1)}^{[ab]}$ and $\xi_{(1)}^{[abc]}$ antisymmetric in tangent indices. The gauge transformations (3.32) are reducible. The corresponding transformations of the gauge parameters $\xi_{(1)}^{[ab]}$ and $\xi_{(1)}^{[abc]}$ read

$$\delta \xi_{(1)}^{[ab]} = \mathcal{D} \chi_{(0)}^{[ab]} + h_c \chi_{(0)}^{[abc]}, \quad \delta \xi_{(1)}^{[abc]} = \mathcal{D} \chi_{(0)}^{[abc]} - 3\lambda^2 h^{[a} \chi_{(0)}^{bc]} \quad (3.33)$$

with the level-2 0-form gauge parameters $\chi_{(0)}^{[ab]}$, $\chi_{(0)}^{[abc]}$ antisymmetric in the tangent indices.

Decompose the 2-form gauge fields (3.29) into

$$e_{\underline{mn}}^{[ab]} \sim \Phi^{[ab],[cd]} \oplus X^{[abc],d} \oplus Y^{[abcd]}, \quad (3.34)$$

$$\omega_{\underline{mn}}^{[abc]} \sim \omega^{[abc],[de]} \oplus Z^{[abcd],e} \oplus W^{[abcde]}, \quad (3.35)$$

where the tensors $\Phi^{[ab],[cd]}$, $X^{[abc],d}$, $\omega^{[abc],[de]}$ and $Z^{[abcd],e}$ have the Young symmetry types $Y(2,2)$, $Y(2,1,1)$, $Y(2,2,1)$ and $Y(2,1,1,1)$, respectively (and contain trace parts). The components $X^{[abc],d}$ and $Y^{[abcd]}$ combine into a single 1-form $\mathcal{E}_{(1)}^{[abc]}$ which can be gauge fixed to zero by the shift generated by the 1-form gauge parameter $\xi_{(1)}^{[abc]}$ (3.32). The remaining component $\Phi^{[ab],[cd]}$ in (3.34) is identified with the dynamical metric-type field and belongs to $B_2^{d-1,1}(2,2)$. Its gauge transformation derived from (3.32) reads

$$\delta \Phi^{[ab],[cd]} = \mathcal{D}^a \xi^{[cd],b} - \mathcal{D}^b \xi^{[cd],a} + \mathcal{D}^c \xi^{[ab],d} - \mathcal{D}^d \xi^{[ab],c}, \quad (3.36)$$

where the parameter $\xi^{[ab],c} \in B_1^{d-1,1}(2,1)$, *i.e.* is antisymmetric in the first two indices, satisfies the Young symmetry condition $\xi^{[ab],c} = 0$ and has non-vanishing trace $\xi^{[ab],b} \neq 0$. In fact, the parameter $\xi^{[ab],c}$ is the component of the 1-form gauge parameter $\xi_{(1)}^{[ab]}$ with its totally antisymmetric part fixed to zero with the aid of the level-2 gauge parameter $\chi_{(0)}^{[abc]}$ (3.33).

The Lorentz-covariant component fields combine into the single 2-form field with $(A)dS_d$ tangent indices as

$$\begin{aligned} \Omega_{(2)}^{[ABC]} &= e_{(2)}^{[ab]} \oplus \omega_{(2)}^{[abc]}, \quad R_{(3)}^{[ABC]} = r_{(3)}^{[ab]} \oplus \mathcal{R}_{(3)}^{[abc]}, \\ \xi_{(1)}^{[ABC]} &= \xi_{(1)}^{[ab]} \oplus \xi_{(1)}^{[abc]}, \quad \chi_{(0)}^{[ABC]} = \chi_{(0)}^{[ab]} \oplus \chi_{(0)}^{[abc]}. \end{aligned} \quad (3.37)$$

The gauge transformations, curvature and Bianchi identities take the form

$$\begin{aligned} \delta \Omega_{(2)}^{[ABC]} &= D_0 \xi_{(1)}^{[ABC]}, \quad \delta \xi_{(1)}^{[ABC]} = D_0 \chi_{(0)}^{[ABC]}, \\ R_{(3)}^{[ABC]} &= D_0 \Omega_{(2)}^{[ABC]}, \quad D_0 R_{(3)}^{[ABC]} = 0, \end{aligned} \quad (3.38)$$

where D_0 is the background AdS_d derivative (2.13), (2.14).

The parity-invariant gauge invariant action is uniquely fixed to the form

$$\mathcal{S}_2 = \frac{1}{\lambda^2} \int_{\mathcal{M}^d} \epsilon_{ABCDEFGM_7 \dots M_{d+1}} E^{M_7} \wedge \dots \wedge E^{M_d} V^{M_{d+1}} \wedge R_{(3)}^{ABC} \wedge R_{(3)}^{DEF}. \quad (3.39)$$

Its variation gives rise to the equations of motion:

$$\frac{\delta \mathcal{S}_2}{\delta \Omega_{(2)}^{[ABC]}} = 0 \quad \Leftrightarrow \quad \epsilon^{ABC}{}_{M_1 \dots M_{d-2}} E^{M_4} \wedge \dots \wedge E^{M_{d-2}} \wedge R_{(3)}^{M_1 M_2 M_3} = 0 . \quad (3.40)$$

Rewriting the equation (3.40) in Lorentz-covariant components one finds

$$\frac{\delta \mathcal{S}_2}{\delta e_{(2)}^{[ab]}} = 0 \quad \Leftrightarrow \quad \mathcal{R}_{(3)}^{[abc]} = h_d \wedge h_e \wedge h_f C^{[abc],[def]} , \quad (3.41)$$

$$\frac{\delta \mathcal{S}_2}{\delta \omega_{(2)}^{[abc]}} = 0 \quad \Leftrightarrow \quad r_{(3)}^{[ab]} = 0 , \quad (3.42)$$

where the 0-form $C^{[abc],[def]}$ is an arbitrary traceless tensor

$$C^{[abc],[def]} \eta_{cd} = 0 \quad (3.43)$$

with the symmetry properties of the two-column Young tableau $Y(2, 2, 2)$. This is the Weyl tensor (2.53).

The equation (3.42) is the constraint on the auxiliary field which expresses it in terms of derivatives of the physical field. Gauge fixing $X^{[abc],d}$ and $Y^{[abcd]}$ to zero (3.34), one solves (3.42) as

$$\omega^{[abc],[de]} = \mathcal{D}^a \Phi^{[bc],[de]} - \mathcal{D}^b \Phi^{[ac],[de]} + \mathcal{D}^c \Phi^{[ab],[de]} , \quad Z^{[abcd],e} = W^{[abcde]} = 0 . \quad (3.44)$$

Substituting this solution into the equation (3.41) which contains first derivatives of the auxiliary field, one finds the second-order equation on the metric-type field $\Phi^{[ab],[cd]}$:

$$(\mathcal{D}^2 + \dots + (d+2)\lambda^2) \Phi^{[ab],[cd]} = 0 , \quad (3.45)$$

where ellipses denote terms containing $\mathcal{D}_c \Phi^{[ca],[bd]}$ and $\Phi^a_{c,[cb]}$. \mathcal{D}^2 is the covariant D'Alembertian given by (3.24). The equation is invariant under the gauge transformations (3.36) and, in the covariant gauge $\Phi^a_{c,[cb]} = 0$, reproduces the equation found by Metsaev [22]. As expected [23], the flat limit of the field equation (3.45) yields no additional symmetries.

Note that the field $\omega^{[abc],[de]}(x)$ (3.35) which appears as $Y(2, 2, 1)$ component of the auxiliary 2-form connection in our formalism, was introduced by Zinoviev in [30]. The Lagrangian of [30] is a particular gauge fixed version of (3.39).

3.3 Two-row rectangular tableaux

Consider now an arbitrary two-row rectangular Lorentz-covariant bosonic field propagating on the flat or $(A)dS_d$ background. The consideration of the present section is in many respects parallel to that of [20, 39] for totally symmetric higher spin fields.

Introduce 2-form gauge field which forms an $(A)dS_d$ tangent tensor described by the three row rectangular Young tableau $Y(s-1, s-1, s-1)$

$$\Omega_{(2)}^{A(s-1), B(s-1), C(s-1)} = dx^{\underline{m}} \wedge dx^{\underline{n}} \Omega_{[\underline{mn}]}^{A(s-1), B(s-1), C(s-1)} , \quad (3.46)$$

subject to the tracelessness condition

$$\Omega_{(2)}^{A(s-1),B(s-1),C(s-1)} \eta_{A(2)} = 0 . \quad (3.47)$$

All other traces are also zero as a consequence of the Young symmetry property. In other words, $\Omega_{(2)} \in B_0^{d-1,2}(s-1, s-1, s-1)$. The decomposition into the Lorentz-covariant higher spin tensors gives

$$\Omega_{(2)}^{A(s-1),B(s-1),C(s-1)} \sim \bigoplus_{t=0}^{s-1} \omega_{(2)}^{a(s-1),b(s-1),c(t)} , \quad (3.48)$$

where all Lorentz Young tableaux on r.h.s. of (3.48) are traceless. In accordance with the prescription of section 2, the Lorentz-covariant 2-form field with zero third row ($t = 0$) is called physical, while those with non-zero third row are auxiliary ($t = 1$) or extra ($t > 1$) and express via derivatives of the physical field by virtue of certain constraints. We do not discuss here the structure of constraints for extra fields as they do not contribute to the free equations of motion. The auxiliary field is expressed in terms of the physical one by virtue of its field equation.

The gauge transformations are

$$\delta \Omega_{(2)}^{A(s-1),B(s-1),C(s-1)} = D_0 \xi_{(1)}^{A(s-1),B(s-1),C(s-1)} \quad (3.49)$$

with the 1-form gauge parameter $\xi_{(1)}$ subject to the same irreducibility conditions as the field $\Omega_{(2)}$. In its turn, the gauge parameter $\xi_{(1)}$ has level-2 gauge transformation

$$\delta \xi_{(1)}^{A(s-1),B(s-1),C(s-1)} = D_0 \chi_{(0)}^{A(s-1),B(s-1),C(s-1)} \quad (3.50)$$

with the level-2 0-form gauge parameter $\chi_{(0)}$.

The linearized higher spin 3-form curvature

$$R_{(3)}^{A(s-1),B(s-1),C(s-1)} = D_0 \Omega_{(2)}^{A(s-1),B(s-1),C(s-1)} \quad (3.51)$$

is invariant under the gauge transformations (3.49).

In accordance with the general formula (2.22) the metric-type field is a part of the physical field

$$\Phi^{a(s),b(s)} = \omega^{[ab];a(s-1),b(s-1)} , \quad (3.52)$$

where the 2-form world indices of the physical field are converted into tangent ones. Other components of $\omega_{(2)}^{a(s-1),b(s-1)} \underline{[mn]}$ can be gauged away with the aid of the shift gauge parameters $\xi_{(1)}^{a(s-1),b(s-1),c}$. As a consequence of the tracelessness condition (3.47) imposed on the (3.46), the field (3.52) satisfies the double-tracelessness conditions

$$\Phi^{a(s),b(s)} \eta_{a(2)} \eta_{b(2)} = 0 . \quad (3.53)$$

Thus $\Phi^{a(s),b(s)} \in B_2^{d-1,1}(s, s)$. From (3.53) it also follows that

$$\Phi^{a(s),b(s)} \eta_{ab} \eta_{b(2)} = 0 , \quad \Phi^{a(s),b(s)} \eta_{a(2)} \eta_{b(2)} + 2 \Phi^{a(s),b(s)} \eta_{ab} \eta_{ab} = 0 . \quad (3.54)$$

These are trace conditions of the work [13], where two-row mixed-symmetry fields on Minkowski space were considered.

The gauge transformation law is

$$\delta\Phi^{a(s),b(s)} = \mathcal{D}^b \Lambda^{a(s),b(s-1)} + (-)^s \mathcal{D}^a \Lambda^{b(s),a(s-1)} . \quad (3.55)$$

Here the gauge parameter Λ is defined as

$$\Lambda^{a(s),b(s-1)} = \xi^{a;a(s-1),b(s-1)} , \quad (3.56)$$

and satisfies the trace conditions

$$\Lambda^{a(s),b(s-1)} \eta_{a(2)} \eta_{a(2)} = 0 , \quad \Lambda^{a(s),b(s-1)} \eta_{b(2)} = 0 . \quad (3.57)$$

Being a consequence of the gauge law (3.49), this definition is consistent with (3.53) and (3.55). In accordance with general consideration of section 2 we see that $\Lambda^{a(s),b(s-1)} \in B_1^{d-1,1}(s, s-1)$.

The metric-type gauge field (3.52) subject to the trace conditions (3.53) is analogous to that considered in [13]. The field trace conditions of [13] arise automatically in our approach as a consequence of the irreducibility of the 2-form in the tangent indices. The difference however is that the gauge parameter is not required to be traceless. Instead, weaker trace conditions (3.57) are imposed, *i.e.* we have more gauge symmetries manifest in our approach.

Let us look for a parity-invariant action in the form

$$\mathcal{S}_2 = \frac{1}{2} \int_{\mathcal{M}^d} \sum_{p=0}^{s-2} a(s, p) \epsilon_{A_1 \dots A_{d+1}} E^{A_7} \wedge \dots \wedge E^{A_d} V^{A_{d+1}} V_{D_1} \dots V_{D_{2(s-p-2)}} \quad (3.58)$$

$$\wedge R_{(3)}^{A_1 B(s-2), A_2 C(s-2), A_3 D(s-2-p) F(p)} \wedge R_{(3)}^{A_4 B(s-2), A_5 C(s-2), A_6 D(s-2-p) F(p)} ,$$

where arbitrary coefficients $a(s, p)$ should be fixed by the extra field decoupling condition. This action makes sense for $d \geq 6$. Its general variation is

$$\begin{aligned} \delta\mathcal{S}_2 = & \frac{(-)^d \lambda}{(d-5)} \int_{\mathcal{M}^d} \sum_{p=0}^{s-2} \left(\frac{(s-p+1)(d-9+2(s-p))}{(s-p-1)} a(s, p) + (s-p-1)a(s, p-1) \right) \\ & \times \epsilon_{A_1 \dots A_{d+1}} E^{A_6} \wedge \dots \wedge E^{A_d} V^{A_{d+1}} V_{D_1} \dots V_{D_{2(s-p)-3}} \\ & \wedge \left(R_{(3)}^{A_1 B(s-2), A_2 C(s-2), D(s-p-1) F(p)} \wedge \delta\Omega_{(2)}^{A_3 B(s-2), A_4 C(s-2), A_5 D(s-p-2) F(p)} \right. \\ & \left. - \delta\Omega_{(2)}^{A_1 B(s-2), A_2 C(s-2), D(s-p-1) F(p)} \wedge R_{(3)}^{A_3 B(s-2), A_4 C(s-2), A_5 D(s-p-2) F(p)} \right) . \end{aligned} \quad (3.59)$$

To impose the extra field decoupling condition one should require all terms in (3.59) to vanish except for that with $p = 0$. This requirement fixes the coefficients $a(s, p)$ up to a normalization factor $\tilde{a}(s)$:

$$a(p, s) = \tilde{a}(s) (-)^p \frac{(s-p)(s-p-1)(d-11+2(s-p))!!}{(s-p-2)!} . \quad (3.60)$$

With these coefficients $a(p, s)$ the action depends essentially only on

$$\Omega_{(2)}^{A(s-1), B(s-1)} \equiv \Omega_{(2)}^{A(s-1), B(s-1), C(s-1)} V_{C(s-1)}, \quad (3.61)$$

which is automatically V -transversal, and on the V -transversal part of

$$\Omega_{(2)}^{A(s-1), B(s-1), C} \equiv \Omega_{(2)}^{A(s-1), B(s-1), CD(s-2)} V_{D(s-2)}. \quad (3.62)$$

These are, respectively, the physical ($t = 0$) and the auxiliary ($t = 1$) fields. The extra fields do not contribute into the free action as guaranteed by the extra field decoupling condition. The auxiliary field $\omega_{(2)}^{a(s-1), b(s-1), c}$ is expressed in terms of the first derivatives of the physical field $\omega_{(2)}^{a(s-1), b(s-1)}$ by virtue of its equation of motion. Insertion of the expression for the auxiliary field $\omega_{(2)}^{a(s-1), b(s-1), c}$ back into the action gives rise to the higher spin action expressed entirely in terms of the metric-type field (3.52) and its first derivatives. The gauge invariance is inbuilt by construction. The flat limit does not yield any additional gauge symmetries. This is in agreement with the general analysis of [23], where the class of Poincare irreps described by rectangular tableaux was argued to admit an AdS_d deformation. Therefore, the resulting action possesses correct higher spin gauge symmetries and describes properly both Minkowski and $(A)dS_d$ free dynamics.

4 Conclusion

The general approach proposed in this paper provides manifestly gauge invariant framework for the formulation of the dynamics of mixed-symmetry massless higher spin gauge fields in (anti) de Sitter and flat space. As demonstrated by the particular examples, the realization of the relevant sets of higher spin fields in terms of p -forms taking values in certain irreducible representations of the $(A)dS$ algebras simplifies analysis considerably and looks promising for the description of the Lagrangian dynamics of a general mixed-symmetry field in $(A)dS_d$. Because higher spin gauge forms introduced in this paper should result from gauging of some non-Abelian higher spin symmetries the proposed approach provides an important information on the structure of underlying higher spin algebras.

Let us note that our approach gives less components for a generic mixed-symmetry metric-type field compared to other examples considered in the literature [12] within local formulation of higher spin dynamics. This phenomenon takes place starting from the first nontrivial example of the field of the symmetry type $Y(3, 2)$. It remains to see whether this indicates some type of reducibility of models of Ref. [12], or is a matter of particular gauge fixing, or is specific for higher spin dynamics in AdS_d . The same time we have more higher spin gauge symmetries manifest compared to some other examples [13].

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